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On a Conjecture of Randić Index and Graph Radius

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Abstract. The *Randić index* $R(G)$ of a graph G is defined as the sum of $(d_i d_j)^{-\frac{1}{2}}$ over all edges $v_i v_j$ of G, where d_i is the degree of the vertex v_i in G. The *radius* $r(G)$ of a graph G is the minimum graph eccentricity of any graph vertex in G. Fajtlowicz in [S. Fajtlowicz, On conjectures of Graffiti, *Discrete Math.* **72** (1988) 113-118] conjectures $R(G) \geq r(G) - 1$ for any connected graph G. A stronger version, $R(G) \geq r(G)$, is conjectured for all connected graphs except even paths by Caporossi and Hansen in [G. Caporossi, et al., Variable neighborhood search for extremal graphs 1: The Autographix system, *Discrete Math.* **212** (2000) 29-44]. In this paper, we make use of *Harmonic index* $H(G)$, which is defined as the sum of $\frac{2}{d_i+d_j}$ over all edges $v_i v_j$ of G, to show that $R(G) \geq r(G) - \frac{31}{105}(k-1)$ for any graph with cyclomatic number $k \geq 1$, and $R(T) > r(T) + \frac{1}{15}$ for any tree except even paths. These results improve and strengthen the known results on these conjectures.

1. Introduction

Topological indices are numerical parameters of a graph which characterize the topological structure of the graph and are usually graph invariants. The *Randić index*, one of the most well-known topological indices, is introduced by Randic [14] and is generalized by Bollobás and Erdös [2]. It studies the branching property of graphs. Since its appearance, tremendously attention has been focused on the upper and lower bounds of the index. Bollobás and Erdös [2] prove that the Randić index of a graph of order n without isolated vertices is at least $\sqrt{n-1}$; they leave the open problem that the minimum value of the Randić index for graphs G with given minimum degree $\delta(G)$. Delorme et al. [4] answer this question for $\delta(G) = 2$, thus partially solve the problem. Furthermore, they prove a best possible lower bound on the Randić index of a triangle-free graph G with given minimum degree $\delta(G)$. Balister et al. [1] build up a technique to determine the maximal Randić index of a tree with a specified number of vertices and leaves. Reviews of mathematical properties of the Randić index refer to [10], [12].

On the other side, Fajtlowicz [7] and Caporossi and Hansen [3] conjecture that the Randić index can be lower bounded in terms of the graph radius. In this paper, we improve and strengthen the known results on these conjectures by studying the relationship between the Harmonic index and graph radius.

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The Harmonic index is defined by Fajtlowicz [8]. Favaron et al. [9] consider the relationship between the Harmonic index and graph eigenvalues. Zhong [16] find the minimum and maximum values of the Harmonic index for simple connected graphs and trees, and characterize the corresponding extremal graphs. Deng et al. [5] consider the relationship between the Harmonic index $H(G)$ and the chromatic number $\chi(G)$ and prove that $\chi(G) \leq 2H(G)$. It strengthens a conjecture of the Randić index and the chromatic number which is based on the system AutoGraphiX and is proved by Hansen and Vukicević [11]. Deng et al. [6] give a best possible lower bound for the Harmonic index of a graph and a triangle-free graph with minimum degree no less than two and characterize the extremal graphs, respectively.

Organization. In Section 2, we introduce neccessary notations used in this paper, and state our main results. In Section 3, 4 and 5, we prove the main results. We conclude our work in Section 6.

2. Preliminary

Let G be a simple undirected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. Let's denote edge $v_iv_j \in E$ if v_i and v_j are adjacent in graph G. Let d_i be the degree of vertex v_i, $i = 1, \ldots, n$. Unless otherwise specified we focus on non-empty connected graph throughout the paper. A pendant vertex is a vertex of degree one. A path with even (odd) vertices is called an even (odd) path. A cycle with even (odd) vertices is called an even (odd) cycle. The neighborhood $N(v_i)$ is the set of vertices adjacent to v_i. The distance $\rho(v_i, v_j)$ is the number of edges in a shortest path connecting v_i and v_j in G. The *radius* of a graph G is the minimum eccentricity of any vertex; that is, $r(G) = \min_{v_i \in V} \max_{v_j \in V} \rho(v_i, v_j)$. The *cyclomatic number* k of a graph G, also known as the *circuit rank*, is the minimum number of edges to remove from the graph to make it cycle-free; that is, $k = |E| - |V| + 1$. Obviously, the cyclomatic number of unicyclic, bicyclic and tricyclic graphs are 1, 2 and 3, respectively.

The celebrated *Randić index* of graph G is introduced by Randić [14].

Definition 2.1. *Given any graph G, the Randić index of G is*

$$R(G) = \sum_{v_iv_j \in E} \frac{1}{\sqrt{d_id_j}},$$

where the sum is over all edges v_iv_j of the graph G.

The following interesting conjecture is proposed by Fajtlowicz [7].

Conjecture 2.2 ([7]). *For any connected graph G, $R(G) \geq r(G) - 1$.*

Caporossi and Hansen [3] prove that $R(T) \geq r(T) + \sqrt{2} - \frac{3}{2}$ for any tree T, and $R(T) \geq r(T)$ for any tree T except even paths. Liu and Gutman [13], and You and Liu [15] prove that the conjecture is true for unicyclic, bicyclic and tricyclic graphs. Caporossi and Hansen [3] also propose the following stronger version of the conjecture.

Conjecture 2.3 ([3]). *For any connected graph G except even paths, $R(G) \geq r(G)$.*

We confirm that the conjectures are true for some graphs by studying the relationship between Harmonic index and graph radius. The *Harmonic index* is defined by Fajtlowicz [8] as follows.

Definition 2.4. *Given any graph G, the Harmonic index of G is*

$$H(G) = \sum_{v_iv_j \in E} \frac{2}{d_i + d_j},$$

where the sum is over all edges v_iv_j of the graph G.

For any path P_n with $n \geq 3$ vertices, it is easy to check that $H(P_n) = \frac{n}{2} - \frac{1}{6}$ and $r(P_n) = \lfloor \frac{n}{2} \rfloor$. Therefore,

$$H(P_n) = \begin{cases} r(P_n) - \frac{1}{6}, & \text{if } n \text{ is even;} \\ r(P_n) + \frac{1}{3}, & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

Since $\sqrt{xy} \leq \frac{x+y}{2}, \forall x, y \in \mathbb{R}^+$, we obtain $R(G) \geq H(G)$ for any graph G.

Our main results have the following three aspects, which improve and strengthen the known results on Conjectures 2.2 and 2.3.

1. For all trees T except even paths, $H(T) > r(T) + \frac{1}{15} > r(T)$.

   We thus partially confirm Conjecture 2.3 and improve the result of [3] for trees.

2. For all unicyclic graphs G, $H(G) \geq r(G)$. The equality holds if and only if G is an even cycle.

   We thus confirm the Conjecture 2.3 for unicyclic graphs.

3. For all graphs G with cyclomatic number $k \geq 1$, $H(G) \geq r(G) - \frac{31}{105}(k-1)$. In particular, $H(G) > r(G) - 1$ for all graphs with cyclomatic number no more than 4.

   We thus confirm that Conjecture 2.2 is true not only for trees, unicyclic, bicyclic and tricyclic graphs, but also for graphs have cyclomatic number 4. In addition, this result implies the inequality in Conjecture 2.2 strictly holds for all graphs with cyclomatic number no more than 4.

3. The Harmonic Index and the Radius of a Tree

We first show that adding a pendant edge to a graph G strictly increases its Harmonic index.

Lemma 3.1. *If G_0 is obtained by adding a pendant edge $v_i v_{n+1}$ to a graph G, where $v_i \in V(G)$, then $H(G_0) > H(G)$.*

Proof. Note that $\frac{1}{x+1} - \frac{1}{x}$ is increasing in $x > 1$. According to the definition of the Harmonic index, we have

$$\begin{aligned} H(G_0) - H(G) &= \frac{2}{d_i + 1 + d_{n+1}} + \sum_{v_j \in N(v_i) \setminus \{v_{n+1}\}} \left(\frac{2}{d_i + 1 + d_j} - \frac{2}{d_i + d_j} \right) \\ &\geq \frac{2}{d_i + 2} + d_i \left(\frac{2}{d_i + 2} - \frac{2}{d_i + 1} \right) \\ &= \frac{2}{(d_i + 1)(d_i + 2)} > 0. \end{aligned}$$

□

Theorem 3.2. *For all trees T except even paths, $H(T) > r(T) + \frac{1}{15}$.*

Proof. Without loss of generality, assume the diameter of the tree T is $k - 1$, and P_k is the diametrical path of T. So $r(T) = r(P_k)$. There are two cases to consider.

1. if k is odd, according to (1), $H(P_k) = r(P_k) + \frac{1}{3}$. In addition, the tree T can be derived from path P_k by adding pendent edges step by step. According to Lemma 1,

   $$H(T) > H(P_k) = r(P_k) + \frac{1}{3} = r(T) + \frac{1}{3} > r(T) + \frac{1}{15}.$$

2. if k is even, according to (1), $H(P_k) = r(P_k) - \frac{1}{6}$. Let the tree T_0 be a subgraph of T, and is obtained by adding one pendent edge to P_k but retaining its diameter; that is, the newly added pendent edge is not incident to the pendent vertices of P_k.

- If the newly added pendent edge is adjacent to the pendent edges of P_k, then by simple calculation we get

$$H(T_0) = H(P_k) + \frac{7}{30}.$$

- If the newly added pendent edge is not adjacent to the pendent edges of P_k, then by simple calculation we get

$$H(T_0) = H(P_k) + \frac{3}{10}.$$

In all, $H(T_0) \geq H(P_k) + \frac{7}{30}$. By the same argument in case 1, we derive the tree T from T_0 by adding pendent edges step by step and get

$$H(T) > H(T_0) \geq H(P_k) + \frac{7}{30} = r(P_k) - \frac{1}{6} + \frac{7}{30} = r(T) + \frac{1}{15}.$$

□

4. The Harmonic Index and the Radius of a Unicyclic Graph

In this section, we discuss the Harmonic index and the radius of unicyclic graphs.

Theorem 4.1. *For all unicyclic graphs G, $H(G) \geq r(G)$. The equality holds if and only if G is an even cycle.*

Proof. Let $C = u_1u_2\cdots u_lu_1$ be the unique cycle of G, where $l \geq 3$, and $|V(G)| = n$. If $G = C$ is a cycle, then $H(G) = \frac{n}{2}$, $r(G) = \lfloor \frac{n}{2} \rfloor$. So $H(G) \geq r(G)$ and the equality holds if and only if n is even. In the following we assume $V(G - C) \neq \emptyset$. Then $T = G - u_iu_{i+1}$ is a spanning tree of G for any edge u_iu_{i+1} of C, and $r(T) \geq r(G)$. We study the following cases.

Case 1. Any longest path of T contains all $l-1$ edges of $C - u_iu_{i+1}$, $\forall u_iu_{i+1} \in E(C)$.

Note that in this case the degree of any vertex of C is at least three. Otherwise there must exist an edge in C such that one of its vertices has degree 2, and other has degree greater than 2. Without loss of generality, assume u_1u_2 is such an edge, where $d_1 = 2, d_2 \geq 3$. Then there must exist a vertex w adjacent to u_2 other than u_1 and u_3. For $C - u_1u_l$, any longest path of T should contain $u_1u_2\cdots u_l$. However, since the length of $wu_2u_3\cdots u_l$ is the same as the length of $u_1u_2u_3\cdots u_l$, there must exist a longest path contains $wu_2u_3\cdots u_l$ which does not include edge u_1u_2. Thus a contradiction occurs.

Without loss of generality, in the following we assume the edge deleted from C is u_1u_l, i.e., $T = G - u_1u_l$. Let $P = v_1v_2\cdots v_t$ denote a longest path of T, where $3 \leq l < t$. So $r(T) = r(P)$ and P contains all $l-1$ edges of $C - u_1u_l$. Note that it is impossible for $l = t$, i.e., $u_l \neq v_t$. For contradiction, suppose $u_l = v_t$. So v_t is a vertex in C with degree no less than 3. Then besides u_1v_t and $v_{t-1}v_t$, there should be another edge connecting v_t. It implies that P is not the longest path; the longest path could be one more edge longer than P. For the same reason, it is impossible for $u_1 = v_1$. In another words, the path $u_1u_2\cdots u_l$ is neither on the leftmost, nor on the rightmost of the longest path P.

Now let's add pendant edges $u_2u_2', u_3u_3', \cdots, u_{l-1}u_{l-1}'$ to P. Denote $T_1 = P + u_2u_2'$, then

$$H(T_1) = H(P) + \frac{3}{10}.$$

Let $T_2 = T_1 + u_3u_3' + \cdots + u_{l-1}u_{l-1}'$. By applying Lemma 1 iteratively, we obtain $H(T_2) > H(T_1)$. Now, let's denote $G' = T_2 + u_1u_l$. By calculation, we get

$$H(G') = H(T_2) + \begin{cases} -\frac{2}{15}, & \text{if } u_1 = v_2 \text{ and } u_l = v_{t-1}; \\ -\frac{1}{15}, & \text{if } u_1 = v_2 \text{ and } u_l = v_{t-2}, \\ & \text{or } u_1 = v_3 \text{ and } u_l = v_{t-1}; \\ 0, & \text{otherwise.} \end{cases}$$

In all, we have

$$H(G') \geq H(T_2) - \frac{2}{15} > H(T_1) - \frac{2}{15} = H(P) + \frac{3}{10} - \frac{2}{15} = H(P) + \frac{1}{6}.$$

Finally, we add all the residual edges $E(G \setminus G')$ to G', step by step. Note that all of these edges are pendant edges. According to Lemma 1, we have $H(G) > H(G')$. Hence, $H(G) > H(G') > H(P) + \frac{1}{6}$. Since P is a path, according to (1), $H(P) \geq r(P) - \frac{1}{6}$. Therefore,

$$H(G) > H(P) + \frac{1}{6} \geq r(P) - \frac{1}{6} + \frac{1}{6} = r(P) = r(T) \geq r(G).$$

Case 2. There is an edge u_iu_{i+1} of C such that $T = G - u_iu_{i+1}$ has a longest path $P = v_1v_2 \cdots v_t$ which contains at most $l-2$ edges of $C - u_iu_{i+1}$.

Obviously, $r(T) = r(P)$. There are three subcases to consider.

(i) P and C have no common vertex.

Let $P_0 = w_1w_2 \cdots w_p$ be the shortest path connecting P and C, where $w_1 = v_i$ is a vertex on P and w_p is a vertex on C, where $2 \leq i \leq t-1$ since P is a longest path of T. Without loss of generality, we assume $w_p = u_1$. Let $T_1 = P + w_1w_2$, then $H(T_1) \geq H(P) + \frac{1}{6}$. Let $T_2 = T_1 + w_2w_3 + \cdots + w_{p-1}w_p$, then $H(T_2) > H(T_1)$ by Lemma 1. Let $T_3 = T_2 + u_1u_2$, then $H(T_3) \geq H(T_2) + \frac{1}{2}$. Let $T_4 = T_3 + u_1u_2 + \cdots + u_{l-1}u_l$, then $H(T_4) > H(T_3)$ by Lemma 1. Now, let $G' = T_4 + u_lu_1$, then $H(G') \geq H(T_4) + \frac{1}{30}$. So, $H(G') > H(P) + \frac{1}{6}$. Finally, let's add all the residual edges $E(G \setminus G')$ to G' step by step. According to Lemma 1, we have $H(G) > H(G')$. Hence,

$$H(G) > H(P) + \frac{1}{6} \geq r(P) = r(T) \geq r(G).$$

(ii) P and C have exactly one common vertex.

Without loss of generality, we assume that $v_i = u_1$ is the unique common vertex of P and C, where $2 \leq i \leq t-1$ since P is a longest path of T. Let $T_1 = P + u_1u_2$, then $H(T_1) \geq H(P) + \frac{1}{6}$. Let $T_2 = T_1 + u_2u_3$, then $H(T_2) \geq H(T_1) + \frac{17}{30} \geq H(P) + \frac{22}{30}$. Let $T_3 = T_2 + u_3u_4 + \cdots + u_{l-1}u_l$, then $H(T_3) > H(T_2)$ by Lemma 1. Now, let $G' = T_3 + u_lu_1$, then $H(G') \geq H(T_3) - \frac{1}{10}$. So, $H(G') > H(P) + \frac{19}{30}$. Finally, let's add all the residual edges $E(G \setminus G')$ to G' step by step. According to Lemma 1, we have $H(G) > H(G')$. Therefore,

$$H(G) > H(P) + \frac{19}{30} > r(P) = r(T) \geq r(G).$$

(iii) $P = v_1v_2 \cdots v_t$ contains $s-1$ edges of C, where $2 \leq s \leq l-1$.

Without loss of generality, we assume that $u_1u_2 \cdots u_s = v_iv_{i+1} \cdots v_{i+s-1}$ for some $1 \leq i < i+s-1 \leq t$, i.e., P contains the edges u_ju_{j+1} $(1 \leq j \leq s-1)$ of C. Then $u_1 \neq v_1$ or $u_s \neq v_t$ since P is a longest path of T. We assume that $u_1 \neq v_1$. Let $T_1 = P + u_su_{s+1}$, then $H(T_1) \geq H(P) + \frac{7}{30}$. Let $T_2 = T_1 + u_{s+1}u_{s+2} + \cdots + u_{l-1}u_l$, then $H(T_2) > H(T_1)$ by Lemma 1. Now, let $G' = T_2 + u_1u_l$, then $H(G') \geq H(T_2) - \frac{1}{30}$ in all cases. So, $H(G') > H(P) + \frac{7}{30} - \frac{1}{30} = H(P) + \frac{1}{5} > r(P)$. Finally, let's add all the residual edges $E(G \setminus G')$ to G' step by step. According to Lemma 1, we have $H(G) > H(G')$. Hence,

$$H(G) > r(P) = r(T) \geq r(G).$$

□

5. The Harmonic Index and the Radius of a Graph with Cyclomatic Number *k*

We will need the following lemmas to prove our main result in this section.

Lemma 5.1. *Let* $f(x,y) = \frac{4}{x} - \frac{8}{x+1} + \frac{2}{x+2} + \frac{4}{y} - \frac{8}{y+1} + \frac{2}{y+2} + \frac{2}{x+y}$, $x, y \in \mathbb{N}^+ \setminus \{1\}$, *then* $f(x,y) \geq -\frac{31}{105}$.

Proof. We first show that $f(x,y) \geq f(5,5) = -\frac{31}{105}$, when $x, y \in \mathbb{R}$ and $x \geq 5, y \geq 5$. Since

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= -\frac{4}{x^2} + \frac{8}{(x+1)^2} - \frac{2}{(x+2)^2} - \frac{2}{(x+y)^2},\\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{4}{(x+y)^3} > 0,\end{aligned}$$

we know that $\frac{\partial f}{\partial x}$ is increasing in y. So,

$$\frac{\partial f(x,y)}{\partial x} \geq \frac{\partial f(x,5)}{\partial x} = -\frac{4}{x^2} + \frac{8}{(x+1)^2} - \frac{2}{(x+2)^2} - \frac{2}{(x+5)^2} = \frac{2(6x^5 + 21x^4 - 96x^3 - 527x^2 - 680x - 200)}{x^2(x+1)^2(x+2)^2(x+5)^2}.$$

Denote $g(x) = 6x^5 + 21x^4 - 96x^3 - 527x^2 - 680x - 200$. It is easy to check that $g(x) > 0$ for $x \geq 5$. Hence, $\frac{\partial f(x,y)}{\partial x} \geq \frac{\partial f(x,5)}{\partial x} > 0$ for $x \geq 5$, which implies $f(x,y)$ is increasing in $x \geq 5$.

Similarly, $f(x,y)$ is increasing in $y \geq 5$. Hence, $f(x,y) \geq f(5,5) = -\frac{31}{105}$.

Second, we compare the values of $f(x,y)$ at several discrete points, namely $f(2,2) = \frac{1}{6}, f(2,3) = -\frac{1}{30}, f(2,4) = -\frac{1}{10}, f(2,5) = -\frac{9}{70}, f(3,3) = -\frac{1}{5}, f(3,4) = -\frac{26}{105}, f(3,5) = -\frac{37}{140}, f(4,4) = -\frac{17}{60}, f(4,5) = -\frac{92}{315}, f(5,5) = -\frac{31}{105}$. Obviously, the minimum is $f(5,5) = -\frac{31}{105}$.

In all, $f(x,y) \geq -\frac{31}{105}$ when $x, y \in \mathbb{N}^+ \setminus \{1\}$. □

Lemma 5.2. *Let G be a graph with cyclomatic number $k \geq 1$, and v_iv_{i+1} is an edge in a cycle of G, then $H(G) \geq H(G - v_iv_{i+1}) - \frac{31}{105}$.*

Proof. According to the definition of the Harmonic index,

$$H(G) - H(G - v_iv_{i+1}) = \frac{2}{d_i + d_{i+1}} + \sum_{v_j \in N(v_i)\setminus\{v_{i+1}\}} \left(\frac{2}{d_i + d_j} - \frac{2}{d_i + d_j - 1}\right) + \sum_{v_j \in N(v_{i+1})\setminus\{v_i\}} \left(\frac{2}{d_{i+1} + d_j} - \frac{2}{d_{i+1} + d_j - 1}\right).$$

Since v_iv_{i+1} is an edge in a cycle, there is a vertex v_{i-1} adjacent to v_i in the cycle with degree $d_{i-1} \geq 2$, and a vertex v_{i+2} adjacent to v_{i+1} in the cycle with degree $d_{i+2} \geq 2$. So,

$$\begin{aligned}H(G) - H(G - v_iv_{i+1}) \geq& \frac{2}{d_i + d_{i+1}} + \left(\frac{2}{d_i + 2} - \frac{2}{d_i + 1}\right) + (d_i - 2)\left(\frac{2}{d_i + 1} - \frac{2}{d_i}\right)\\ &+ \left(\frac{2}{d_{i+1} + 2} - \frac{2}{d_{i+1} + 1}\right) + (d_{i+1} - 2)\left(\frac{2}{d_{i+1} + 1} - \frac{2}{d_{i+1}}\right)\\ =& \frac{4}{d_i} - \frac{8}{d_i + 1} + \frac{2}{d_i + 2} + \frac{4}{d_{i+1}} - \frac{8}{d_{i+1} + 1} + \frac{2}{d_{i+1} + 2} + \frac{2}{d_i + d_{i+1}}\\ \geq& -\frac{31}{105} \quad \text{(Lemma 5.1)}.\end{aligned}$$

□

Theorem 5.3. *Let G be a graph with cyclomatic number $k \geq 1$. Then $H(G) \geq r(G) - \frac{31}{105}(k-1)$. In particular, $H(G) > r(G) - 1$ for a graph with cyclomatic number no more than 4.*

Proof. We first prove the case $k \geq 2$. Since the cyclomatic number of graph G is $k \geq 2$, there exists a sequence of edges $e_1, \ldots, e_k$ such that the cyclomatic number of graph $G_i = G - \{e_1, e_2, \cdots, e_i\}$ is $k - i$, where $1 \leq i \leq k$. In particular, G_{k-1} is a spanning unicylic subgraph of G. Note that $r(G) \leq r(G_1) \leq r(G_2) \leq \cdots \leq r(G_{k-1})$. According to Lemma 5.2 and Theorem 4.1, we have

$$\begin{aligned}H(G) \geq& H(G_1) - \frac{31}{105} \geq H(G_2) - \frac{31}{105} - \frac{31}{105} \geq \cdots \geq H(G_{k-1}) - \frac{31}{105}(k-1)\\ \geq& r(G_{k-1}) - \frac{31}{105}(k-1) \geq r(G) - \frac{31}{105}(k-1).\end{aligned}$$

Note that this is in accord with Theorem 4.1. That is, when G is unicyclic graph, i.e., $k = 1$, $H(G) \geq r(G)$. Therefore, the theorem is true for $k \geq 1$.

In particular, $H(G) \geq r(G) - 1$ for $1 \leq k \leq 4$. □

6. Conclusion

We focus on the conjectures of Randić index and graph radius. These conjectures have been opened for a long time. We improve and strengthen the known results on the conjectures by studing the relationship between the Harmonic index and graph radius. It is interesting to know whether or not the conjectures are true for more general graphs. In particular, could the techniques in this paper be extended to studying more general graphs? It is intriguing to know whether the following conjecture is true.

Conjecture 6.1. *For all connected graphs G except even paths, $H(G) \geq r(G)$.*

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